# ELECTROMAGNETIC RAYS

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**Introduction.** Max Jammer informs us[1] that speculation about the nature of light, when traced to its pre-Socratic and Judeo-Christian beginnings, becomes entangled with speculation about the nature of God on the one hand[2] and the nature of space on the other.

Since God/space, however conceived, are profoundly non-local—manage to be "everywhere at once"—one might assert only semi-facetiously that optics came into the world as a field theory. But in point of fact its non-theological aspects sprang from attempts to understand the mechanisms of vision. The early notion that vision is accomplished by "tentacles" that radiate from the eye was replaced in time by the inverse notion that vision results from "rays" that fall upon the eye. In his *Optics* (*c.* 300 BC) he develops the geometry of "lines of sight," discusses how such normally straight lines are altered by the presence of reflective surfaces and refractive media, and constructs a theory of perspective. The latter theory was lost, then reinvented in the early Italian Renaissance by Leon Alberti (1404–1472) and (independently) by Piero della Francesa (1412–1492).[3]

Interest in the geometry of rays passed in time from painters to natural philosophers, who were concerned especially with the geometry of rays in more complex (refractive) environments. Snell's law (unpublished work dated 1621) —which had been stated already in 984 by Ibn Sahl and in 1602 by Thomas Harriot[4]—was independently rederived by René Descartes and used by him

[1] *Concepts of Space* (1960), pages 34–38.

[2] The identification of light with the Sun, and hence with the gods, can be traced back even farther, to the very beginnings of recorded thought.

[3] An excellent brief account of this pretty subject has been prepared under the direction of the mathematician Helmer Aslaksen [http://www.math.nus.edu.sg/∼mathelmr/] and can be found at http://www.math.nus.edu.sg/aslaksen/projects/perspective/home2.htm.

[4] See http://en.wikipedia.org/wiki/Snell's_law.

(1637) to construct a theory or the rainbow, a theory already anticipated by a certain Persian astronomer by about 1300 and by Theodoric of Freiberg in 1307.[5] In 1662 Pierre de Fermat put forth a variational principle (the "Principle of Least Time—a sweeping generalization of the variational principle advanced by Hero of Alexandria in about 60 AD—that served in principle to describe all rays, whether they are subjected to reflective or refractive manipulation.[6]

Whether light involves the "motion of something" (as Empedocles (490–430 BC) had maintained, and as Fermat's language would appear to require), how briskly that "something" moves[7] , and of what might be the nature of that "something" are questions that had been pondered by natural philosophers for at least two thousand years by the time (1676) Ole Rømer managed (by an astrophysical technique) to *measure* the speed of light: using Rømer's data and his own estimate of the length of an astronomical unit (AU), Christiaan Huygens obtained the value 136,000 miles per second, which is 26% lower than the presently accepted value. By 1728 James Bradley, using a different astronomical technique that laid to rest any lingering possibility that the speed might be infinite, had obtained a figure that is only 1% too small.

Those developments failed, however, to indicate what might be the physical nature of the "moving somethings" that constitute light. In 1678—proceeding independently along lines that had been contemplated a few years earlier by Robert Hooke and that might be considered to be consonant with Descartes' opinion that light was "a disturbance in the plenum"—Huygens developed a wave theory of light (published as *Treatise on Light* in 1690) from which, in particular, Fermat's principle could be obtained as a corollary. Newton, on the other hand, insisted (1675) that the fact that "light travels in a straight line" is accounted for most simply by the assumption that light rays are by nature streams of "corpuscles."[8] His influental *Optiks* (1704) provided a summary account of the theory which Newton erected on the basis of his corpuscular assumption.

By 1750 it had become obvious to Euler and to a growing number of others that the phenomena of interference and diffraction—whether optical or acoustic —are most easily explained by a wave theory, and the force of this observation

[5] See http://en.wikipedia.org/wiki/Rainbow.

[6] See http://en.wikipedia.org/wiki/Fermat's_principle. Hero's principle had things to say only about reflected rays, while Fermat's principle provided a unified account of the effects of reflection and refraction.

[7] Descartes argued that the speed of light must necessarily be infinite, else his "entire system of philosophy would collapse": Aristotle and Kepler were—for their own reasons—in agreement with Descarte.

[8] The young Newton had first encountered this notion in a posthumous publication of Pierre Gassendi, and may have been attracted to it partly because it is so contrary to the view advanced by Descartes, to whose philosophy he tended generally to take strong exception (as, indeed, did Gassendi).

was made inescapable by the work (early 19th Century) of Thomas Young (1773–1829),[9] which inspired the sophisticated work of Augustin-Jean Fresnel (1788–1827). Fresnel's contributions to the theory of diffraction are well known: less well known is the fact that in 1821 Fresnel, enlarging upon another of the ideas first put forward by Young, established by mathematical argument that to account for what was then known about optical polarization phenomena (which have no acoustic analog) the optical vibration must be purely transverse (have no longitudinal component). Notable contributions to the mathematical theory of diffraction were made also by—among many others—George Gabriel Stokes (1819–1903), who in 1852 brought the formal theory of optical polarization to very nearly its present state of perfection—remarkably, since he had no idea *what it was* that was vibrating transversally!

That essential information was supplied by Maxwell (1831–1879), who discovered in 1861 that the equations (newly developed by him) that describe electromagnetic fields possess—amazingly, and very much to his surprise—wavelike solutions that propagate with precisely the speed of light, and who wrote in 1864 that "light [appears on this evidence to be] an electromagnetic disturbance, propagated ... according to electromagnetic laws."[10]

Maxwell had made it possible at last to trace the rich phenomenology of (classical) physical optics to first principles. Or nearly: the concepts and methods he devised to describe the electromagnetic properties of the *materials* with which light interacts in many/most of the contexts of greatest physical interest were entirely phenomenological, and would necessarily remain so until the invention of quantum mechanics. Attempts to understand the interaction of light not with the material substance of (for example) lenses/crystals but with the most elemental of materials—atoms, molecules—would, in turn, require the invention of a quantum electrodynamics, and of its subdiscipline: *quantum optics*.[11]

As was already remarked, practical theory relating to the essentials of physical optics (interference, diffraction, polarization, dispersion) had been brought to a remarkably high state of development even prior to Maxwell's discovery of the electromagnetic nature of light. A parallel but separate line of development—tracing back, as we have seen, to Euclid, and to which Gauss

[9] See http://en.wikipedia.org/wiki/Thomas_Young_ %28Scientist%29. Young—an almost exact contemporary of Beethoven, and a practicing physician for most of his professional life—was a polymath who made fundamental contributions not only to physical optics (two-slit diffraction) but also to the theory of color vision (it was he who first conjectured that the eye contains nerves responsive to red, green and violet, he who first associated "color" with "wavelength"), physiology and medicine, linguistics and the study of Egyptian hieroglyphics, music theory.

[10] See http://en.wikipedia.org/wiki/Light and its links for an excellent survey of the historical material of which I have attempted to provide here merely a short summary.

[11] See http://en.wikipedia.org/wiki/Quantum_optics.

and Hamilton had recently made notable contributions—had by the early 19th Century brought to a comparably high state of perfection *geometrical optics*, which is concerned with the geometry of rays ("ray tracing"), which provides the language of choice for lens makers and the designers of optical devices.

Evidently the continuum theory of light did not displace the ray theory—with which it had been in competition for millennia—but subsumed it. The question I wish to consider is this: How did that come about? How do "light rays" and their established properties (for example: Fermat's Principle of Least Time, Snell's Law, *etc.*) come into being as electromagnetic artifacts?

**1. Electromagnetism in the presence of material media.** The data upon which Maxwell based his theory was obtained from experiments performed in the presence of material media (air, water, castor oil, iron, ...): his theory therefore was, perforce, a theory simultaneously of the electromagnetic field and the electromagnetic properties of simple materials.[12] He arrived at field equations that can be written[13]

$$\left.\begin{aligned} \mathrm{div}\,\mathbf{D} &= 4\pi\rho \\ \mathrm{div}\,\mathbf{B} &= 0 \\ \mathrm{curl}\,\mathbf{H} &= \tfrac{1}{c}\big\{4\pi\mathbf{j} + \tfrac{\partial}{\partial t}\mathbf{D}\big\} \\ \mathrm{curl}\,\mathbf{E} &= \tfrac{1}{c}\big\{\quad \mathbf{0} - \tfrac{\partial}{\partial t}\mathbf{B}\big\} \end{aligned}\right\} \tag{1}$$

These field equations would remain incomplete/insoluable in the absence of the following "constitutive (or material) equations"

$$\left.\begin{aligned} \mathbf{j} &= \sigma\,\mathbf{E} \\ \mathbf{D} &= \varepsilon\,\mathbf{E} \\ \mathbf{B} &= \mu\,\mathbf{H} \end{aligned}\right\} \tag{2}$$

Here $\{\sigma, \varepsilon, \mu\}$ are typical "coefficients of susceptibility." Specifically

- $\sigma$ is the *specific conductivity*;
- $\varepsilon$ is the *dielectric constant* (though not always constant: see below);
- $\mu$ is the *magnetic permeability*.

The materials of primary optical interest (vacuum, air, various liquids and glasses) permit some major simplifications: they are typically non-conductive

[12] It was H. A. Lorentz who, in the 1890s (nearly twenty years after Maxwell's death), first recognized/emphasized the conceptual importance of eliminating the "stuff;" of studying the electromagnetic field itsownself—*in vacuo*. Of course, good vacuums were relative rare commodities in Victorian laboratories.

[13] I adopt the notational/dimensional conventions of J. D. Jackson (*Classical Electrodynamics* (1st ed., 1962), page 177) and M. Born & E. Wolf (*Principles of Optics* (7th ed., 1962), page 1)). To obtain agreement with D. F. Griffiths (*Introduction to Electrodynamics* (3rd ed., 1999), page 330) one must set $4\pi \to 1$ and absorb the $\frac{1}{c}$ factors into the definitions of $\mathbf{B}$, $\mathbf{D}$ and $\mathbf{j}$.

($\sigma = 0$) and present no free charge ($\rho = 0$). In such cases, equations (1) assume the form

$$\left.\begin{aligned} \operatorname{div}(\varepsilon\,\mathbf{E}\,) &= 0 \\ \operatorname{div}(\mu\mathbf{H}) &= 0 \\ \operatorname{curl}\mathbf{H} &= +\tfrac{1}{c}\tfrac{\partial}{\partial t}(\varepsilon\,\mathbf{E}\,) \\ \operatorname{curl}\mathbf{E} &= -\tfrac{1}{c}\tfrac{\partial}{\partial t}(\mu\mathbf{H}) \end{aligned}\right\} \tag{3}$$

The final pair of equations are coupled, but can be "decoupled by differentiation," as I now demonstrate. Proceeding on the assumption that $\varepsilon$ and $\mu$ are time-independent, we introduce the 3rd equation into curl of the final equation and obtain

$$\operatorname{curl}\big(\tfrac{1}{\mu}\operatorname{curl}\mathbf{E}\big) + \big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2(\varepsilon\,\mathbf{E}\,) = \mathbf{0}$$

But[14] the identity $\operatorname{curl}\big(f\,\mathbf{A}\big) = f\operatorname{curl}\mathbf{A} - \mathbf{A}\times(\text{grad f})$ suppl ies

$$\operatorname{curl}\big(\tfrac{1}{\mu}\operatorname{curl}\mathbf{E}\big) = \tfrac{1}{\mu}\operatorname{curl}\operatorname{curl}\mathbf{E} - (\operatorname{curl}\mathbf{E})\times(\operatorname{grad}\tfrac{1}{\mu})$$

which by $\operatorname{curl}\operatorname{curl}\mathbf{A} = \operatorname{grad}(\operatorname{div}\mathbf{A}) - \nabla^2\mathbf{A}$ becomes

$$= \tfrac{1}{\mu}\operatorname{grad}(\operatorname{div}\mathbf{E}) - \tfrac{1}{\mu}\nabla^2\mathbf{E} + \tfrac{1}{\mu}(\operatorname{curl}\mathbf{E})\times(\operatorname{grad}\log\mu)$$

giving

$$\nabla^2\mathbf{E} - \varepsilon\mu\big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2\mathbf{E} + (\operatorname{grad}\log\mu)\times(\operatorname{curl}\mathbf{E}) - \operatorname{grad}(\operatorname{div}\mathbf{E}) = 0 \tag{4}$$

From the first of the equations (3) we obtain $\varepsilon\operatorname{div}\mathbf{E} + \mathbf{E}\cdot\operatorname{grad}\varepsilon = 0$ whence

$$\operatorname{div}\mathbf{E} = -\mathbf{E}\cdot\operatorname{grad}\log\varepsilon$$

which we can use to obtain this variant of (4):

$$\nabla^2\mathbf{E} - \varepsilon\mu\big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2\mathbf{E} + (\operatorname{grad}\log\mu)\times(\operatorname{curl}\mathbf{E}) + \operatorname{grad}(\mathbf{E}\cdot\operatorname{grad}\log\varepsilon) = 0 \tag{5.1}$$

A similar argument supplies

$$\nabla^2\mathbf{H} - \varepsilon\mu\big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2\mathbf{H} + (\operatorname{grad}\log\varepsilon)\times(\operatorname{curl}\mathbf{H}) + \operatorname{grad}(\mathbf{H}\cdot\operatorname{grad}\log\mu) = 0 \tag{5.2}$$

In *homogeneous* media

$$\operatorname{grad}\log\varepsilon = \operatorname{grad}\log\mu = 0$$

[14] See, for example, the front endpaper in Griffiths' *Electrodynamics*.

and equations (5) assume the familiar simple form

$$\nabla^2\,\mathbf{E} - \varepsilon\mu\big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2\mathbf{E} = 0 \tag{6.1}$$

$$\nabla^2\,\mathbf{H} - \varepsilon\mu\big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2\mathbf{H} = 0 \tag{6.2}$$

**PROBLEM 1.** Relax the assumption that the simple homogeneous medium is non-conductive (*i.e.*, that $\sigma = 0$) while retaining the assumption that $\rho = 0$. Show that the argument which led to (6) now leads to the equations

$$\left.\begin{aligned}\nabla^2\,\mathbf{E} - \varepsilon\mu\big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2\mathbf{E} - \frac{4\pi\mu\sigma}{c^2}\tfrac{\partial}{\partial t}\mathbf{E} = 0\\ \nabla^2\,\mathbf{H} - \varepsilon\mu\big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2\mathbf{H} - \frac{4\pi\varepsilon\sigma}{c^2}\tfrac{\partial}{\partial t}\mathbf{H} = 0\end{aligned}\right\} \tag{7}$$

which are close variants of the "telegrapher's equation;" see in this connection http://en.wikipedia.org/wiki/Telegrapher's_equations. Non-zero (but finite) conductivity speaks of non-zero resistivity, and electrical resistance provides a familiar energy-loss mechanism. One is therefore not surprised to discoved that light projected into such a medium is exponentially attenuated (absorbed).

**2. Index of refraction.** We are working within a formalism in which—as follows directly from (1)—$[\mathbf{E}] = [\mathbf{B}]$ and $[\mathbf{D}] = [\mathbf{H}]$. The ratios $\mathbf{B}/\mathbf{E}$ and $\mathbf{H}/\mathbf{D}$ therefore have the same dimension because both are, in fact, dimensionless. But from (2) we obtain $(\mathbf{B}/\mathbf{E}) = \varepsilon\mu(\mathbf{H}/\mathbf{D})$. It follows that the product

$$\varepsilon\mu \text{ is dimensionless}$$

and so also (as follows most directly from (7)) are the products $\varepsilon\sigma$ and $\mu\sigma$.

Equation (6.1) can be written

$$\nabla^2\,\mathbf{E} - \big(\tfrac{1}{v}\tfrac{\partial}{\partial t}\big)^2\mathbf{E} = 0 \tag{8}$$

with

$$v = c/\sqrt{\varepsilon\mu}$$

The identification of $v$ with *phase velocity* arises from the following familiar argument: assume

$$\mathbf{E}\,(\boldsymbol{x},t) = \boldsymbol{\mathcal{E}}\,e^{i(\boldsymbol{k}\cdot\boldsymbol{x}\,-\,\omega t)} \equiv \boldsymbol{\mathcal{E}}\,e^{i(\text{phase})}$$

The wave equation (8) then requires $k^2 - \omega^2/v^2 = 0$, while $\frac{d}{dt}\text{phase} = 0$ gives $\boldsymbol{k}\cdot\dot{\boldsymbol{x}} = \omega$. So

$$\text{phase velocity} \equiv \text{magnitude of } \dot{\boldsymbol{x}} = \omega/k = v$$

The dimensionless *index of refraction* is defined

$$\text{index of refraction} \equiv \frac{\text{phase velocity in vacuum}}{\text{phase velocity in medium}} \quad : \quad n = c/v = \sqrt{\varepsilon\mu}$$

In the materials of relevance to optics it is almost invariably the case that in excellent approximation

$$\mu \approx \mu_0 \quad \text{(vacuum value)}$$

so one can with that same precision write

$$n \approx \sqrt{\varepsilon \mu_0}$$

In vacuum, $v_{\text{phase}} = c$ so $n_0 = 1$, which entails

$$\varepsilon_0 \mu_0 = 1$$

and permits us to write

$$n \approx \sqrt{\varepsilon/\varepsilon_0} = \sqrt{\varepsilon_{\text{relative to vacuum}}}$$

Though we are most familiar with materials in which $v_{\text{phase}} < c$ (and therefore $n > 1$), it is important to appreciate that *this inequality is not a forced implication of relativity*,[15] and that there are contexts (for example, in the design of X-ray telescopes) in which $n < 1$ is of critical importance. Very recently it has been established that one can construct artificial structures in which—remarkably—the index of refraction $n$, is *negative*! [16]

**3. Inhomogeneity, anisotropy and other complications.** Look into a pan of mineral oil that is seen to be gently rolling because gently heated from below. Such a system clearly provides an instance of an index of refraction that varies from point to point, and from time to time:

$$n = n(\boldsymbol{x}, t)$$

Unannealed glass provides a frozen instance of such a system:

$$n = n(\boldsymbol{x}) \quad : \quad \text{typical of "inhomogeneous" media}$$

Such optical inhomogeneity can evidently be attributed to spatial variability of the dielectric constant: $\varepsilon(\boldsymbol{x}) \Rightarrow n(\boldsymbol{x}) = \sqrt{\varepsilon(\boldsymbol{x})\mu_0}$. The simple wave equation (6.1)

[15] Relativity requires that $v_{\text{information}} < c$. But there is, so far as I am aware, no universally applicable precise definition of "$v_{\text{information}}$". It is often interpreted to mean "signal velocity," which in simple contexts can be identified with "group velocity." But people sometimes speak in this connection of the "velocity of energy transport" and "front velocity" (the velocity of the leading edge of a pulse). It is well established that there are circumstances under which $v_{\text{group}} > c$: see http://www.phy.duke.edu/research/photon/qelectron/proj/infv/fast_debate.ptml and the link to Fast Light Tutorial, which provides an animated instance of a superluminal wave packet.

[16] See http://www.sciencemag.org/cgi/content/abstract/292/5514/77.

—which has become

$$\nabla^2 \mathbf{E} - n^2 \big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2 \mathbf{E} = 0$$

—has then to be replaced by this variant[17]

$$\nabla^2 \mathbf{E} - [n(\boldsymbol{x})]^2 \big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2 \mathbf{E} + 2\,\mathrm{grad}\big\{\,\mathbf{E}\cdot\mathrm{grad}\,\log\,[n(\boldsymbol{x})]\big\} = 0 \tag{9}$$

of (5.1).

In inhomogeneous media of the sort considered thus far the "electric vector" $\mathbf{E}$ and the "electric displacement vector" $\mathbf{D}$ stand in the location-dependent relationship

$$\mathbf{D}(\boldsymbol{x}) = \varepsilon(\boldsymbol{x})\,\mathbf{E}(\boldsymbol{x}) \quad : \quad \mathbf{D} \text{ and } \mathbf{E} \text{ are at all points parallel}$$

But crystals—by their very nature, and even in the absence of inhomogeneities —present *anisotropic* environments: the dielectric constant becomes matrix-valued

$$\mathbf{D} = \varepsilon\,\mathbf{E} \quad \text{becomes} \quad \mathbf{D} = \boldsymbol{\varepsilon}\,\mathbf{E}$$

with the consequence that parallelism is, in general, lost. Energy conservation can be shown under quite general conditions[18] to entail that

$$\text{the real } 3\times 3 \text{ matrix } \boldsymbol{\varepsilon} \text{ is } \textit{symmetric}$$

The "dielectric tensor" $\boldsymbol{\varepsilon}$ serves to define three mutually orthogonal *principal directions* in the crystal, and to associate a distinct index of refraction with each.

In *optically active media* the dielectric tensor (matrix) acquires what is in effect an antisymmetric "gyroscopic component":[19]

$$\boldsymbol{\varepsilon} \longmapsto \boldsymbol{\varepsilon} + \mathbf{g}\times$$

The notation here exploits the fact that

$$\boldsymbol{\varepsilon}_{\text{antisymmetric}} \equiv \begin{pmatrix} 0 & -g_3 & g_2 \\ g_3 & 0 & -g_1 \\ -g_2 & g_1 & 0 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \times$$

The distinguishing feature of such media is that the crystal structure—or the structure of the molecules suspended in liquid solution—is distinct from its mirror image. Quartz crystals—which occur naturally in both right- and left-handed forms—are optically active, but fused quartz (which is an amorphous, non-crystaline form of quartz) is not. When linearly polarized light is projected into an optically active medium the polarizational orientation rotates (in either the right- or the left-handed sense, depending upon the medium) through an

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[17] Use $(\mathrm{grad}\,\log\mu_0) = 0$, $(\mathrm{grad}\,\log\varepsilon) = \mathrm{grad}\,\log(\varepsilon\mu_0)$, $\varepsilon\mu_0 = n^2$.

[18] See Born & Wolf, page 791.

[19] See Amnon Yariv & Pochi Yeh, *Optical Waves in Crystals* (1984), page 97.

angle proportion to the thickness of the medium traversed. If reflected back to point of entry the effect is expunged (since a right-hand screw, seen from the wrong end, looks like a left-hand screw). Optical activity can be considered to result from the circumstance that linearly polarized ↕ ight and ↔ light have distinct indices of refraction. Biological materials are almost always optically active, for familiar biomolecular reasons (chirality of amino acids[20]).

Place virtually any optical medium in a strong magnetic field and project linearly polarized light along the magnetic axis. Again one sees orientational rotation proportional to the thickness of the medium traversed. This magneto-optic effect ("Faraday rotation") was discovered by Michael Faraday in 1845, and provided the first evidence that light and electromagnetism must be in some way related. Faraday rotation looks superficially like optical activity, but the underlying mechanism is so different as to have this major consequence: when the Faraday rotated light is reflected back to its point of entry the rotation is not undone, but doubled (so repeated reflection back and forth—as happens, for example, inside a laser—can greatly magnify the effect). The Faraday effect can be considered to result from the fact that circularly polarized ↻ light and ↺ light have distinct indices of refraction, and has many important astrophysical applications.[21]

I discuss now how the important notion of a *complex index of refraction* comes about.[22] At (7) we had

$$\nabla^2 \mathbf{E} - \varepsilon\mu_0 \left(\tfrac{1}{c}\tfrac{\partial}{\partial t}\right)^2 \mathbf{E} - \frac{4\pi\mu_0\sigma}{c^2}\frac{\partial}{\partial t}\mathbf{E} = 0 \tag{10}$$

Assume the $\mathbf{E}$-field to be monochromatic: $\mathbf{E}(\boldsymbol{x},t) = \boldsymbol{\mathcal{E}}(\boldsymbol{x})\, e^{-i\,\omega\, t}$. Then

$$\nabla^2\boldsymbol{\mathcal{E}} + \hat{k}^2\boldsymbol{\mathcal{E}} = \mathbf{0}$$

with

$$\begin{aligned}\hat{k}^2 &\equiv \frac{\omega^2\mu_0}{c^2}\left(\varepsilon + i\,\frac{4\pi\sigma}{\omega}\right)\\ &= \frac{\omega^2\mu_0}{c^2}\hat{\varepsilon} \quad \text{with} \quad \hat{\varepsilon} \equiv \varepsilon + i\,\frac{4\pi\sigma}{\omega} \equiv \text{"complex dielectric coefficient"}\end{aligned}$$

We now write

$$\hat{n} \equiv \sqrt{\hat{\varepsilon}\mu_0} = \frac{c\,\hat{k}}{\omega} = \frac{c}{\hat{v}}$$

which give back familiar relationships among real variables when the hats are removed. To make explicit the complexity of $\hat{n}$, people often write

$$\hat{n} = n(1 + i\kappa)$$

and call the real parameter $\kappa$ the "attenuation index" of "extinction coefficient."

[20] See http://en.wikipedia.org/wiki/Chirality_%28chemistry%29.

[21] See http://en.wikipedia.org/wiki/Faraday_effect.

[22] The following discussion has been taken from Born & Wolf's §14.1 "Wave propagation in a conductor, " pages 735–739.

**PROBLEM 1.** Use the results just established to show that

$$n^2(1-\kappa^2) = \varepsilon\mu_0$$

$$n^2\kappa = \frac{2\pi\mu_0\sigma}{\omega} = \frac{\mu_0\sigma}{\nu}$$

$$n^2 = \tfrac{1}{2}\left(\sqrt{\varepsilon^2\mu_0^2 + \frac{4\mu_0^2\sigma^2}{\nu^2}} + \varepsilon\mu_0\right) \tag{11.1}$$

$$n^2\kappa^2 = \tfrac{1}{2}\left(\sqrt{\varepsilon^2\mu_0^2 + \frac{4\mu_0^2\sigma^2}{\nu^2}} - \varepsilon\mu_0\right) \tag{11.2}$$

Indicate what becomes of these equations in the non-conductive limit $\sigma \downarrow 0$.

As we have seen, when non-zero conductivity is taken into account—forcing the wave equation

$$\nabla^2\,\mathbf{E} - \varepsilon\mu\left(\tfrac{1}{c}\tfrac{\partial}{\partial t}\right)^2\mathbf{E} = 0$$

to assume the modified form

$$\nabla^2\,\mathbf{E} - \varepsilon\mu\left(\tfrac{1}{c}\tfrac{\partial}{\partial t}\right)^2\mathbf{E} - \frac{4\pi\mu\sigma}{c^2}\tfrac{\partial}{\partial t}\,\mathbf{E} = 0$$

—the familiar plane wave solution[23] $\mathbf{E}\,(\boldsymbol{x},t) = \boldsymbol{\mathcal{E}}\,e^{i(k\boldsymbol{s}\cdot\boldsymbol{x} - \omega t)}$ goes over into

$$\mathbf{E}\,(\boldsymbol{x},t) = \boldsymbol{\mathcal{E}}\,e^{i(\hat{k}\boldsymbol{s}\cdot\boldsymbol{x} - \omega t)}$$

with $\hat{k} = \omega\,\hat{n}/c = \omega\,n(1+i\kappa)/c = k(1+i\kappa) \equiv k + \frac{1}{2}i\alpha$.[24] We therefore have

$$= e^{-\frac{1}{2}\alpha\boldsymbol{s}\cdot\boldsymbol{x}}\cdot\boldsymbol{\mathcal{E}}\,e^{i(k\boldsymbol{s}\cdot\boldsymbol{x} - \omega t)}$$

which describes a plane wave that is attenuated as it progresses. Energy goes as amplitude squared, so the

$$\text{energetic attenuation factor} = e^{-\alpha\boldsymbol{s}\cdot\boldsymbol{x}}$$

diminishes with a characteristic distance (or "depth") given by

$$d = \frac{1}{\alpha} = \frac{1}{2k\kappa} = \frac{1}{4\pi\kappa}\lambda = \frac{1}{4\pi\,n\kappa}\lambda_0$$

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[23] Here it has served clarity to write $\boldsymbol{k} = k\boldsymbol{s}$, with $\boldsymbol{s}$ a dimensionless unit vector.

[24] Here

$$k = \frac{n\omega}{c} = nk_0 = \frac{2\pi}{\lambda}$$

where $\lambda \equiv$ wavelength in material $= \dfrac{\text{wavelength in vacuum}}{n} \equiv \lambda_0/n$.

In metals, $\sigma$ is typically so large that $\mu_0\sigma/\nu \gg \varepsilon\mu_0$ for frequencies that are not too high ($\nu < \nu_{\text{optical}}$). Equations (11) can then be approximated

$$n \approx n\kappa \approx \sqrt{\frac{\mu_0\sigma}{\nu}} \tag{12}$$

giving

$$d \approx \frac{\lambda_0}{4\pi}\sqrt{\frac{\nu}{\mu_0\sigma}} = \sqrt{\frac{\nu\lambda_0^2}{16\pi^2\mu_0\sigma}} = \sqrt{\frac{c^2}{16\pi^2\mu_0\sigma\nu}} = \frac{c}{\sqrt{8\pi\mu_0\sigma\omega}}$$

Evidently light cannot penetrate at all into superconductors ($d \downarrow 0$ as $\sigma \uparrow \infty$).

The preceding discussion illuminates why sheets of glass are transparent, sheets of metal are opaque (though dim light may pass through very thin foils). It teaches us that attempts to develop a "transparent but electrically conductive glass are foredoomed to failure, but fails to account for the fact that some non-conductive materials (sheets of rubber, for example) are opaque.

We have seen that in conductive media the real and imaginary parts of $\hat{n}$ becomes frequency-dependent.[25] More generally, since the microscopic parts (atoms, molecules) of bulk materials possess charcteristic resonance properties (both individually and by virtue of the way they are assembled), we expect the microdynamical response of such systems to harmonic electromagnetic stimulation to depend upon the frequency of the incident radiation. We expect therefore that the dielectric constant will in all realistic cases be a function of frequency (and also of temperature, *etc*, and that so also will be the real and imaginary parts of the index of refraction:

$$\hat{n} \longmapsto \hat{n}(\nu)$$

Which is, of course, well known to be in fact the case: we touch here on the reasons that prisms are dispersive, lenses tend to display chromatic aberration, and Prof. Powell is able to gain useful information by studying the absorption spectra of DNA samples. Though theoretical attempts to account for the observed $\nu$-dependence of $\hat{n}$ are necessarily quantum mechancal, useful insight can be gained from elementary classical analysis.[26]

[25] At (12) we had $\hat{n} = n + in\kappa = (1+i)\sqrt{\mu_0\sigma/\nu}$.
[26] See §14.3 in Born & Wolf, pages 749–752.